JOURNAL OF APPROXIMATION THEORY 78, 174–189 (1994)

# Interpolatory Hermite Spline Wavelets

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*Communicated by Charles A. Micchelli*

Received July 22, 1992; accepted in revised form April 28, 1993

Wavelets are constructed comprising spline functions with multiple knots. These wavelets have certain derivatives vanishing at the integers, in an analogous manner to the $B$-splines of Schoenberg and Sharma related to cardinal Hermite interpolation. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

We do not attempt to give here a review of the development of the theory of wavelets, but refer to [2, 4, 11, 12]. Although the theory extends to more than one dimension, we restrict our attention here to the univariate case.

Let $\psi$ be a function in $L^2(\mathbf{R})$ and consider its translated dilates $B := \{2^{k/2}\psi(2^k - j): j, k \in Z\}$. We call $\psi$ an *orthogonal wavelet* if $B$ forms an orthonormal basis for $L^2(\mathbf{R})$. We call $\psi$ a *wavelet* (sometimes called prewavelet) if $B$ forms a Riesz basis for $L^2(\mathbf{R})$ and $\psi(2^k. - j)$ is orthogonal to $\psi(2^l. - i)$ whenever $k \neq l$. (A set $\{\phi_j : j \in Z\}$ in $L^2(\mathbf{R})$ is a Riesz basis for $L^2(\mathbf{R})$ if every function $f$ in $L^2(\mathbf{R})$ can be expressed uniquely in the form $\sum_{-\infty}^{\infty} c_j\phi_j$ and the norm $\|f\| := \|c\|_2$ is equivalent to the norm $\|f\|_2$). The weaker notion of wavelet was considered more recently than that of orthogonal wavelet, see [1, 8], and is particularly useful in allowing the construction of compactly supported spline wavelets [3].

In [6, 7], this concept is weakened further, as follows. We say functions $\psi_0, ..., \psi_{r-1}$ are wavelets of multiplicity $r$ if $B := \{2^{k/2}\psi_s(2^k. - j): j, k \in Z, s = 0, ..., r-1\}$ forms a Riesz basis for $L^2(\mathbf{R})$ and $\psi_s(2^k. - j)$ is orthogonal to $\psi_t(2^k. - i)$ whenever $k \neq l$. In [7], this idea is used to construct compactly supported spline wavelets $\psi_0, ..., \psi_{r-1}$ with knots of multiplicity $r$, which are analogous to consecutive $B$-splines with knots of multiplicity $r$.

In this paper we give a different construction of spline wavelets $\psi_0, ..., \psi_{r-1}$ with knots of multiplicity $r$, which are analogous to the



0021-9045/94 $6.00

$B$-splines introduced by Schoenberg and Sharma [14], which are related to the problem of cardinal Hermite spline interpolation. Here each wavelet $\psi_s$, $0 \leqslant s \leqslant r-1$, satisfies the interpolation conditions

$$\psi_s^{(j)}(k) = 0, \qquad 0 \leqslant j \leqslant r-1, j \neq s, k \in Z.$$

Thus data values on the derivatives of order $s$ at the integers are picked up only by integer translates of the wavelet $\psi_s$, and not by integer translates of the wavelets $\psi_j$, $j \neq s$.

The construction of the wavelets $\psi_0, ..., \psi_{r-1}$ is given in Section 2 and their properties are studied in Section 3. The work here depends heavily on the work of Lee [9] in showing that the $B$-splines are locally linearly independent, and on the theory of cardinal Birkhoff interpolation in [5]. Finally, in Section 4, we examine the special case of cubic splines with double knots, and in this case relate the wavelets of this paper with those of [7].

## 2. Construction of Wavelets

We denote by $\zeta_{n,r}(S)$ the space of spline functions of degree $n$ on $\mathbf{R}$ with knots of multiplicity $r$ on the set $S$. For $i = 0, ..., r-1$, we let $N_i$ denote the $B$-spline in $\zeta_{2r-1,r}(Z)$ with support on $[0, 2]$ and knots at 0, 1, and 2 of multiplicity $r-i$, $r$, and $i+1$, respectively, (with suitable normalisation). Then any function $f$ in $\zeta_{2r-1,r}(Z)$ can be written uniquely in the form

$$f = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{r-1} a_{ij} N_j(. - i)$$

for numbers $(a_{ij})$.

Instead of this usual basis of $B$-splines for $\zeta_{2r-1,r}(Z)$, we shall consider an alternative basis introduced by Schoenberg and Sharma [14] and shown to be a basis by Lee in [9]. For $s = 0, ..., r-1$, we let $B_s$ denote the unique element of $\zeta_{2r-1,r}(Z)$ with support on $[0, 2]$ and satisfying

$$B_s^{(j)}(1) = \delta_{sj}, \qquad j = 0, ..., r-1. \tag{2.1}$$

Now $B_0, ..., B_{r-1}$ form a basis for $\zeta_{2r-1,r}(Z) \mid [0, 2]$ and hence $N_0, ..., N_{r-1}$ can be written as linear combinations of $B_0, ..., B_{r-1}$. It follows that any function $f$ in $\zeta_{2r-1,r}(Z)$ with support in $[k, k+N]$ for $k$ in $Z$ and $N \geqslant 2$ can be written in the form

$$f = \sum_{i=k}^{k+N-2} \sum_{j=0}^{r-1} a_{ij} B_j(. - i), \tag{2.2}$$

where by (2.1),

$$a_{ij} = f^{(j)}(i+1).$$

In particular, we see that since $\zeta_{2r-1,r}(Z) \subset \zeta_{2r-1,r}(\frac{1}{2}Z)$, we have

$$B_s(x) = \sum_{i=0}^{2} \sum_{j=0}^{r-1} c_{ij} B_j(2x-i), \qquad s = 0, ..., r-1, \tag{2.3}$$

where

$$c_{ij} = 2^{-j} B_s^{(j)}\left(\frac{i+1}{2}\right).$$

We remark that the basis $(B_j)$ is defined for degree $2m-1$ for any $m \geqslant r$, but it is only for degree $n = 2r-1$ that we are able to express any function in $\zeta_{n,r}(Z)$ of compact support as a finite linear combination as in (2.2).

Now let $V_0 = \zeta_{2r-1,r}(Z) \cap L^2(\mathbf{R})$, $V_1 = \zeta_{2r-1,r}(\frac{1}{2}Z) \cap L^2(\mathbf{R})$ and let $W$ be the orthogonal complement of $V_0$ in $V_1$. It is known [7] that $\{N_j(.-i)$: $i \in Z$, $j = 0, ..., r-1\}$ forms a Riesz basis for $V_0$. Since $N_0, ..., N_{r-1}$ and $B_0, ..., B_{r-1}$ are equivalent bases, it follows that $\{B_j(.-i)$: $i \in Z$, $j = 0, ..., r-1\}$ is also a Riesz basis for $V_0$. The two-scale relation (2.3) suggests that we look for wavelets $\psi_s$ corresponding to the $B$-splines $B_s$, as we now describe.

For $s = 0, ..., r-1$ define

$$T_s = \{f \in V_1 : f^{(j)} \mid Z = 0, 0 \leqslant j \leqslant r-1, j \neq s\}.$$

For even $r$ and $s = 0, ..., r-1$, we shall construct a function $\psi_s$ in $W \cap T_s$ with support on $[0, r+2]$ so that $\{\psi_s(.-i)$: $i \in Z$, $s = 0, ..., r-1\}$ forms a Riesz basis for $W$. It then follows from the work of [6] that $\psi_0, ..., \psi_{r-1}$ are wavelets of multiplicity $r$, as defined in Section 1. To do this we consider, for $s = 0, ..., r-1$, the space

$$U_s = \{f \in \zeta_{4r-1,r}(\tfrac{1}{2}Z) : f^{(j)} \mid Z = 0, 0 \leqslant j \leqslant r-1, 2r \leqslant j \leqslant 3r-1, j \neq 2r+s\}.$$

We also define

$$U = \{f \in \zeta_{4r-1,r}(\tfrac{1}{2}Z) : f^{(j)} \mid Z = 0, j = 0, ..., r-1\}.$$

By integrating by parts it is easy to see that we have

LEMMA 2.1. *If $f$ in $W \cap T_s$ has support in $[a, b]$, $a < b$, then there is a unique function $g$ in $U_s$ with support in $[a, b]$ and $g^{(2r)} = f$. Conversely if $g$ in $U_s$ has support in $[a, b]$, then $g^{(2r)}$ is in $W \cap T_s$.*

We shall construct functions $\Psi_s$ in $U_s$, $s=0, ..., r-1$, and then define $\psi_s = \Psi_s^{(2r)}$.

Consider the function

$$S(x) = \sum_{r}^{2r-1} a_j x^j + \sum_{3r}^{4r-1} a_j x^j + \sum_{3r}^{4r-1} b_j (x-\tfrac{1}{2})_+^j, \qquad 0 \leqslant x \leqslant 1, \quad (2.4)$$

and for $\lambda$ in $\mathbf{R}$ consider the equations

$$\begin{cases} S^{(j)}(1) = 0, j=0, ..., r-1, & j=2r, ..., 3r-1, \\ S^{(j)}(1) - \lambda S^{(j)}(0) = 0, & j=r, ..., 2r-1. \end{cases} \quad (2.5)$$

This gives a homogeneous system of $3r$ equations in the unknowns $a_r, ..., a_{2r-1}, a_{3r}, ..., a_{4r-1}, b_{3r}, ..., b_{4r-1}$. We denote the determinant of this system by $\pi(\lambda)$.

Now take $s$, $0 \leqslant s \leqslant r-1$. For $S$ as in (2.4), consider the function

$$T(x) = S(x) + c\frac{x^{2r+s}}{(2r+s)!}, \qquad 0 \leqslant x \leqslant 1. \quad (2.6)$$

For $\lambda$ in $\mathbf{R}$ and $0 \leqslant t \leqslant 1$, we consider the equations

$$T^{(j)}(1) = 0, \qquad j=0, ..., r-1, 2r, ..., 2r+s-1, \quad (2.7)$$

$$T^{(2r+s)}(1) - \lambda T^{(2r+s)}(0) = 0, \quad (2.8)$$

$$T^{(j)}(1) = 0, \qquad j=2r+s+1, ..., 3r-1, \quad (2.9)$$

$$T^{(j)}(1) - \lambda T^{(j)}(0) = 0, \qquad j=r, ..., 2r-1, \quad (2.10)$$

$$T(t) = 0. \quad (2.11)$$

This gives a homogeneous system of $3r+1$ equations in the $3r$ previous unknowns together with the unknown $c$. We denote its determinant by $\pi_s(\lambda, t) = \pi_s(t)$. Since $T^{(2r+s)}(0) = c$, we have

$$\pi_s^{(2r+s)}(0) = \pi(\lambda). \quad (2.12)$$

For example, when $r=1$,

$$\pi_0(\lambda, t) = \begin{vmatrix} 1 & 1 & \frac{1}{8} & \frac{1}{2} \\ 0 & 6 & 3 & 1-\lambda \\ 1-\lambda & 3 & \frac{3}{4} & 1 \\ t & t^3 & (t-\frac{1}{2})_+^3 & \frac{1}{2}t^2 \end{vmatrix}.$$

Considering (2.11), (2.7), and (2.9) gives, for general $r$,

$$\pi_s^{(j)}(0) = \pi_s^{(j)}(1) = 0, \qquad 0 \leqslant j \leqslant r-1, 2r \leqslant j \leqslant 3r-1, j \neq 2r+s, \quad (2.13)$$

while (2.11), (2.8), and (2.10) give

$$\pi_s^{(j)}(1) = \lambda \pi_s^{(j)}(0), \qquad j = r, \ldots, 2r-1 \text{ and } 2r+s. \tag{2.14}$$

From (2.13) and (2.14) we see that $\pi_s(t)$ can be extended to an element $\pi_s$ of $U_s$ satisfying

$$\pi_s(t+1) = \lambda \pi_s(t), \qquad t \in \mathbf{R}. \tag{2.15}$$

We now write

$$\pi_s(\lambda, t) = \sum_{k=0}^{r+1} \Phi_{s,k}(t)\, \lambda^{r+1-k}, \qquad 0 \leqslant t \leqslant 1, \tag{2.16}$$

and define

$$\begin{aligned} \Psi_s(t) &:= \Phi_{s,k}(t-k), \qquad k \leqslant t < k+1,\ k = 0, \ldots, r+1, \\ &:= 0, \qquad \text{otherwise}. \end{aligned} \tag{2.17}$$

Equating coefficients of powers of $\lambda$ in (2.13) and (2.14) gives

$$\begin{gathered} \Phi_{s,k}^{(j)}(0) = \Phi_{s,k}^{(j)}(1) = 0, \\ k = 0, \ldots, r+1,\ 0 \leqslant j \leqslant r-1,\ 2r \leqslant j \leqslant 3r-1,\ j \neq 2r+s, \end{gathered} \tag{2.18}$$

$$\Phi_{s,k}^{(j)}(1) = \Phi_{s,k+1}^{(j)}(0), \qquad k = 0, \ldots, r,\ j = r, \ldots, 2r-1 \text{ and } 2r+s, \tag{2.19}$$

$$\Phi_{s,0}^{(j)}(0) = \Phi_{s,r+1}^{(j)}(1) = 0, \qquad j = r, \ldots, 2r-1 \text{ and } 2r+s. \tag{2.20}$$

From (2.17)–(2.20) we see that $\Psi_s$ lies in $U_s$. Clearly from (2.17), $\Psi_s$ has support in $[0, r+2]$. So by Lemma 2.1, the function $\psi_s = \Psi_s^{(2r)}$ is in $W \cap T_s$ and has support in $[0, r+2]$.

To finish this section we note that by (2.15)–(2.17),

$$\begin{aligned} \pi_s(t) &= \sum_{k=-\infty}^{\infty} \Psi_s(t+k)\, \lambda^{r+1-k}, \qquad t \in \mathbf{R}, \\ &= \sum_{k=-\infty}^{\infty} \Psi_s(t-k)\, \lambda^{r+1+k}, \qquad t \in \mathbf{R}, \end{aligned} \tag{2.21}$$

while by (2.12),

$$\begin{aligned} \pi(\lambda) &= \sum_{k=1}^{r+1} \Psi_s^{(2r+s)}(k)\, \lambda^{r+1-k} \\ &= \sum_{k=0}^{r} \Psi_s^{(2r+s)}(r+1-k)\, \lambda^k. \end{aligned} \tag{2.22}$$

## 3. Properties of Wavelets

We now study properties of the functions $\psi_0, ..., \psi_{r-1}$, in particular showing that $\{\psi_s(.-i): i \in Z, s = 0, ..., r-1\}$ forms a Riesz basis for $W$ and hence $\psi_0, ..., \psi_{r-1}$ are wavelets of multiplicity $r$. As in the previous section, we shall first consider the functions $\Psi_0, ..., \Psi_{r-1}$ which, by (2.17), is equivalent to studying the functions $\{\Phi_{s,k}\}$ given by (2.16). Henceforward we assume that $r$ is even.

Lemma 3.1. *For* $0 \leqslant s \leqslant r-1$ *and any real number* $\lambda$, *the function* $\pi_s = \pi_s(\lambda, .)$ *does not vanish identically on* $\mathbf{R}$.

*Proof.* We shall apply the theory of [5]. Since $\pi(\lambda)$ is the determinant of the system (2.5), the roots of $\pi(\lambda) = 0$ are the eigenvalues for the following cardinal Birkhoff interpolation problem.

$$\left.\begin{array}{l}\text{Find a function } f \text{ in } \zeta_{4r-1,r}(\tfrac{1}{2}Z) \text{ with prescribed values for} \\ f^{(j)}(k),\ k \in Z, j \in I,\end{array}\right\} \tag{3.1}$$

where $I = \{0, ..., r-1, 2r, ..., 3r-1\}$. We shall apply a special case of Theorem 4.6 of [5], which we now state. For a problem of form (3.1), let $J = \{r \leqslant j \leqslant 4r-1 : 4r-1-j \notin I\}$. Suppose that $J = \{j_1, ..., j_r\}$, where $j_1 < \cdots < j_r$, and for some $\rho$, $\eta$,

$$j_k + k + r + \eta \quad \text{is} \quad \begin{cases} \text{odd} & \text{if } \ 1 \leqslant k \leqslant \rho, \\ \text{even} & \text{if } \ \rho + 1 \leqslant k \leqslant r. \end{cases}$$

Then (3.1) has $\rho$ distinct eigenvalues of sign $(-1)^{\eta}$ and $r - \rho$ distinct eigenvalues of sign $(-1)^{\eta+1}$.

For the case above we have $J = \{2r, ..., 3r-1\}$ and, since $r$ is even, there are $r$ distinct, strictly positive eigenvalues. Moreover, by symmetry, the eigenvalues are invariant under $t \to t^{-1}$ and so they are not equal to 1.

Now the values of $\lambda$ for which $\pi_s^{(r)}(0) = \pi_s^{(r)}(\lambda, 0) = 0$ are the eigenvalues for the cardinal Birkhoff interpolation problem (3.1) with $I = \{0, ..., r, 2r, ..., 3r-1\} \setminus \{2r+s\}$.

In this case $J = \{2r-1-s, 2r, ..., 3r-2\}$ and as above we see that if $s$ is even, then the $r$ eigenvalues are distinct, strictly negative and not equal to $-1$, while if $s$ is odd, the eigenvalues comprise 1 and $r-1$ distinct strictly negative eigenvalues, including $-1$.

So if $\lambda \leqslant 0$ or $\lambda = 1$, then from (2.12),

$$\pi_s^{(2r+s)}(0) = \pi(\lambda) \neq 0,$$

while if $\lambda > 0$, $\lambda \neq 1$, then $\pi_s^{(r)}(0) \neq 0$. So for all real $\lambda$, $\pi_s$ does not vanish identically. ∎

A similar argument shows that Lemma 3.1 is true for $r$ odd and $s$ even. Unfortunately, however, it does not hold when both $r$ and $s$ are odd, for in this case $\pi_s(-1, .)$ vanishes identically. For $r$ and $s$ odd, arguing as in the proof of Lemma 3.1 shows that for $\lambda = -1$, $\pi_s^{(2r+s)}(0) = \pi_s^{(r)}(0) = 0$ and considering a finite Birkhoff interpolation problem on any large enough interval shows that $\pi_s$ must vanish on this interval.

LEMMA 3.2. *For* $0 \leqslant s \leqslant r-1$, *the functions* $\Phi_{s,i}$, $i = 0, ..., r+1$, *are linearly independent on* $[0, \frac{1}{2}]$ *and on* $[\frac{1}{2}, 1]$.

*Proof.* This follows closely the proof of Lemma 1 in [9]. Suppose that

$$\sum_{i=0}^{r+1} a_i \Phi_{s,i}(x) = 0, \qquad \tfrac{1}{2} \leqslant x \leqslant 1,$$

for some constants $(a_i)$. By (2.20) we have

$$\sum_{i=0}^{r} a_i \Phi_{s,i}^{(j)}(1) = 0, \qquad j = r, ..., 2r-1 \text{ and } 2r+s.$$

This gives $r+1$ equations in $r+1$ unknowns. Let $\Delta$ denote the determinant of this system:

$$\Delta := \det[\Phi_{s,i}^{(j)}(1)].$$

We shall show that $\Delta \neq 0$. It follows that $a_0 = \cdots = a_r = 0$. Since $\Phi_{s,r+1}(t) = \pi_s(0, t)$, this does not vanish identically, by Lemma 3.1, and so we also have $a_{r+1} = 0$. This shows that $\Phi_{s,0}, ..., \Phi_{s,r+1}$ are linearly independent on $[\frac{1}{2}, 1]$ and the result for $[0, \frac{1}{2}]$ follows similarly.

Now let $\lambda_1, ..., \lambda_r$ be the roots of $\pi(\lambda) = 0$, which we showed in the proof of Lemma 3.1 are distinct and strictly positive. Letting $\lambda_0$ be any non-zero value distinct from $\lambda_1, ..., \lambda_r$, put

$$V := \det[\lambda_j^{r+1-i}]_{i,j=0}^{r}.$$

Then

$$\begin{aligned} \Delta V &= \det\left[\sum_{k=0}^{r} \Phi_{s,k}^{(j)}(1)\, \lambda_i^{r+1-k}\right] \\ &= \det[\pi_s^{(j)}(\lambda_i, 1)], \end{aligned}$$

by (2.16) and (2.20). By (2.15) and (2.12),

$$\pi_s^{(2r+s)}(\lambda_i, 1) = \lambda_i \pi_s^{(2r+s)}(\lambda_i, 0) = \lambda_i \pi(\lambda_i).$$

Since $\pi(\lambda_i)=0$, $i=1, ..., r$, we have

$$\Delta V=(-1)^r \lambda_0 \pi(\lambda_0) \det[\pi_s^{(j)}(\lambda_i, 1)]_{i=1}^{r} {}_{j=r}^{2r-1}.$$

Since $\lambda_0 \pi(\lambda_0) \neq 0$, we only need to show that

$$\det[S_i^{(j)}(1)]_{i=1}^{r} {}_{j=r}^{2r-1} \neq 0, \tag{3.2}$$

where we have written

$$S_i(t)=\pi_s(\lambda_i, t), \qquad t \in \mathbf{R}.$$

For $i=1, ..., r$, $S_i$ does not vanish identically, by Lemma 3.1, and by (2.15),

$$S_i(t+1)=\lambda_i S_i(t), \qquad t \in \mathbf{R}.$$

Moreover by (2.12) and (2.13),

$$S_i^{(j)}(k)=0, \qquad k \in Z, j=0, ..., r-1, 2r, ..., 3r-1.$$

In the terminology of [13, 5], $S_1, ..., S_r$ are eigensplines for the problem (3.1). Now suppose that

$$\sum_{i=1}^{r} c_i S_i^{(j)}(1)=0, \qquad j=r, ..., 2r-1,$$

and let

$$\begin{aligned} S(x) &= 0, \qquad x \leqslant 1, \\ &= \sum_{i=1}^{r} c_i S_i(x), \qquad x \geqslant 1. \end{aligned}$$

Then $S$ lies in $\zeta_{4r-1,r}(\frac{1}{2}Z)$ and

$$S^{(j)}(k)=0, \qquad k \in Z, i=0, ..., r-1, 2r, ..., 3r-1.$$

So from the theory of [5], $S$ is a linear combination of the eigensplines. Since the eigensplines are linearly independent on $(-\infty, 0)$, we must have $S \equiv 0$ and hence $\sum_{i=1}^{r} c_i S_i \equiv 0$ on $(1, \infty)$. Since the eigensplines are linearly independent on $(1, \infty)$ we must have $c_i=0$, $i=1, ..., r$. Thus (3.2) is established and the proof is complete. ∎

Lemma 3.2 tells us, in particular, that none of the functions $\Phi_{s,0}, ..., \Phi_{s,r+1}$ can vanish identically on $[0, \frac{1}{2}]$ or on $[\frac{1}{2}, 1]$ and so definition (2.17) immediately gives

COROLLARY 3.1. *For $0 \leqslant s \leqslant r-1$, the function $\Psi_s$ does not vanish identically on any nontrivial interval in $[0, r+2]$.*

LEMMA 3.3. *For $0 \leqslant s \leqslant r-1$, any function $f$ in $U_s$ can be written uniquely in the form*

$$f = \sum_{i=-\infty}^{\infty} c_i \Psi_s(.-i) \tag{3.3}$$

*for some constants $(c_i)$. Moreover there is a constant $K$ such that for any $f$ in $U_s$ and any integer $j$,*

$$|c_i| \leqslant K \, \|f \mid [j, j+1]\|_\infty, \qquad i = j-r-1, ..., j. \tag{3.4}$$

*Proof.* Consider the following interpolation problem. Find $g$ in $\zeta_{4r-1, r}(\frac{1}{2}Z)[0, 1]$ with prescribed values for

$$\begin{cases} g^{(j)}(0), & j = 0, ..., 3r-1, \\ g^{(j)}(1), & j = 0, ..., r-1, 2r, ..., 3r-1. \end{cases} \tag{3.5}$$

This is a problem of quasi-Hermite interpolation by Hermite splines and it follows from standard theory [10] that it has a unique solution for all choices of data. Thus for $0 \leqslant s \leqslant r-1$, the space $U_s \mid [0, 1]$ has dimension $r+2$. But by (2.18) the functions $\Phi_{s,i}$, $i = 0, ..., r+1$, lie in $U_s \mid [0, 1]$ and, by Lemma 3.2, they form a basis for $U_s \mid [0, 1]$. Now by (2.17),

$$\Phi_{s,i}(t) = \Psi_s(t+i), \qquad 0 \leqslant t \leqslant 1, \; i = 0, ..., r+1,$$

and thus for $f$ in $U_s$ we can write uniquely

$$f(x) = \sum_{i=0}^{r+1} c_i \Psi_s(x+i), \qquad 0 \leqslant x \leqslant 1. \tag{3.6}$$

Considering again the interpolation problem (3.5), we see that the space

$$\zeta_s := \{ g \in U_s \mid [0, 1] : g^{(j)}(0) = 0, j = r, ..., 2r-1, 2r+s \}$$

has dimension 1. But by (2.20), $\Phi_{s,0}$ lies in $\zeta_s$ and so forms a basis for $\zeta_s$. Now let

$$f_1(x) := f(x) - \sum_{i=0}^{r+1} c_i \Psi_s(x+i), \qquad x \in \mathbf{R}. \tag{3.7}$$

By (3.6), $f_1$ vanishes on $[0, 1]$ and so $f_1(.+1)$ lies in $\zeta_s$. Thus there is a unique constant $c_{-1}$ so that

$$\begin{aligned} f_1(x+1) &= c_{-1} \Phi_{s,0}(x), & 0 \leqslant x \leqslant 1, \\ &= c_{-1} \Psi_s(x), & 0 \leqslant x \leqslant 1, \end{aligned}$$

by (2.17). So by (3.7) we can write uniquely

$$f(x) = \sum_{i=-1}^{r+1} c_i \Psi_s(x+i), \qquad 0 \leqslant x \leqslant 2.$$

Continuing in this manner for increasing and decreasing $x$ gives (3.3).

To prove (3.4) we take any integer $j$ and note that $\Psi_s(.-i) \mid [j, j+1]$, $i = j-r-1, ..., j$, form a basis for $U_s[j, j+1]$. Since norms on a finite dimensional space are equivalent, there is a constant $K$ such that for all $f$ in $U_s$,

$$\max\{|c_i| : j-r-1 \leqslant i \leqslant j\} \leqslant K \|f \mid [j, j+1]\|_\infty .$$

Since $K$ is clearly independent of $j$, this completes the proof. ∎

THEOREM 3.1. *Any bounded function $f$ in $U$ can be written uniquely in the form*

$$f = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \Psi_s(.-i),$$

*for uniformly bounded constants $c_i^{(s)}$. Moreover, if $f(x)$ decays exponentially as $|x| \to \infty$, then $c_i^{(s)}$ decays exponentially as $|i| \to \infty$, $s = 0, ..., r-1$.*

*Proof.* Consider again the cardinal Birkhoff interpolation problem (3.1). From the theory of [5] this problem is "solvable," i.e., for bounded date there is a unique bounded solution and if the data decays exponentially as $|j| \to \infty$, then the solution decays exponentially as $|x| \to \infty$.

It follows that we can write any bounded function $f$ in $U$ in the form $f = \sum_{s=0}^{r-1} g_s$, where for $s = 0, ..., r-1$, $g_s$ is bounded and lies in $U_s$. Moreover, if $f(x)$ decays exponentially as $|x| \to \infty$, then for $s = 0, ..., r-1$, $g_s(x)$ decays exponentially as $|x| \to \infty$.

The result now follows from Lemma 3.3. ∎

So far in this section we have derived properties of the functions $\Psi_0, ..., \Psi_{r-1}$. We shall now deduce properties of the wavelets $\psi_s = \Psi_s^{(2r)}$, $s = 0, ..., r-1$. Recall that $\psi_s$ lies in $W \cap T_s$ and has support in $[0, r+2]$.

THEOREM 3.2. *Take $0 \leqslant s \leqslant r-1$. Any element of $W \cap T_s$ with support in $[0, r+2]$ is a constant multiple of $\psi_s$. The function $\psi_s$ does not have support on any interval $[a, b]$ strictly in $[0, r+2]$ and for any integer $j$, $0 \leqslant j \leqslant r+1$, $\psi_s$ does not vanish identically on $[j, j+1]$. Moreover $\psi_s$ is either symmetric or anti-symmetric about $r/2+1$.*

*Proof.* Suppose that $g$ is an element of $W \cap T_s$ with support in $[0, r+2]$. Then by Lemma 2.1, there is a function $f$ in $U_s$ with support in $[0, r+2]$ satisfying $f^{(2r)} = g$. By Lemma 3.3, $f$ can be expressed in the form (3.3). Applying Lemma 3.2 on the interval $[-1, 0]$ gives $c_i = 0$, $-r-2 \leqslant i \leqslant -1$. Similarly applying it on $[r+2, r+3]$ gives $c_i = 0$, $1 \leqslant i \leqslant r+2$. Thus the restriction of $f$ to $[0, r+2]$ equals $c_0 \Psi_s$ and since $f$ has support on $[0, r+2]$, we have $f = c_0 \Psi_s$. Hence $g = c_0 \psi_s$.

If $\psi_s$ has support on an interval $[a, b]$ strictly in $[0, r+2]$, then by Lemma 2.1, $\Psi_s$ also has support on $[a, b]$ which contradicts Corollary 3.1.

Next suppose that $\psi_s$ vanishes identically on $[j, j+1]$ for some integer $j$, $0 \leqslant j \leqslant r+1$. Then we can write $\psi_s = F + G$, where $F$ has support in $[0, j]$ and $G$ has support in $[j+1, r+2]$. By the previous part of the result, $\psi_s$ cannot vanish identically on $[0, 1]$ and so $F$ cannot vanish identically. Clearly $F$ is in $T_s$. We claim that $F$ lies in $W$. For $i \geqslant j$ and $k = 0, ..., r-1$, $B_k(.-i)$ vanishes on $[0, j]$ and so $\int FB_k(.-i) = 0$. Next consider $i \leqslant j-1$. Then for $k = 0, ..., r-1$, $B_k(.-i)$ vanishes on $[j+1, r+2]$ and so $\int GB_k(.-i) = 0$. Since $\psi_s$ is in $W$, $\int (F+G) B_k(.-i) = 0$ and so we again have $\int FB_k(.-i) = 0$. Since $\{B_k(.-i): i \in Z, k = 0, ..., r-1\}$ forms a basis for $V_0$, $F$ is orthogonal to $V_0$, i.e. $F$ lies in $W$. So $F$ is an element of $W \cap T_s$ with support in $[0, j]$, which contradicts the two earlier parts of the result.

Finally, we note that $\psi_s(r+2-.)$ is an element of $W \cap T_s$ with support in $[0, r+2]$ and so $\psi_s(r+2-.) = c\psi_s$, where $\psi_s = c^2 \psi_s$ and so $c = \pm 1$. ∎

We say a sequence $(f_i)_{-\infty}^{\infty}$ of functions is *locally linearly independent* on an interval $(a, b)$ if whenever $\sum_{-\infty}^{\infty} c_i f_i$ vanishes identically on $(a, b)$, then $c_i = 0$ for all $i$ for which $f_i$ does not vanish identically on $(a, b)$.

THEOREM 3.3. *For $0 \leqslant s \leqslant r-1$ and any integer $j$, the sequence $(\psi_s(.-i))_{i=-\infty}^{\infty}$ is locally linearly independent on $(j, j+1)$.*

*Proof.* Without loss of generality we may assume $j = 0$. Suppose that $f = \sum_{-\infty}^{\infty} c_i \psi_s(.-i)$ vanishes identically on $(0, 1)$. Let $g = \sum_{-r-1}^{0} c_i \psi_s(.-i)$. Then $f$ coincides with $g$ on $(0, 1)$ and so $g$ vanishes identically on $(0, 1)$. Then $g = g_1 + g_2$, where $g_1$ has support in $[-r-1, 0]$ and $g_2$ has support in $[1, r+2]$. Clearly $g_1$ and $g_2$ are in $T_s$. By the same argument as in the last part of the proof of Theorem 3.2, $g_1$ and $g_2$ are in $W$. So by Theorem 3.2, $g_2$ is a constant multiple of $\psi_s$ and, as $g_2$ vanishes on $[0, 1]$, it must vanish identically. Similarly, $g_1$ vanishes identically and hence $g$ vanishes identically.

On $[r+1, r+2]$, $g$ coincides with $c_0 \psi_s$ and so $c_0 = 0$. Continuing in this way gives $c_{-1} = \cdots = c_{-r-1} = 0$. Thus the sequence $(\psi_s(.-i))_{i=-\infty}^{\infty}$ is locally linearly independent on $(0, 1)$. ∎

*Remark.* The sequence $(\psi_s(.-i))_{i=-\infty}^{\infty}$ is *not* locally linearly independent on $(0, \frac{1}{2})$. To see this we note that $W \cap T_s \mid (0, \frac{1}{2})$ lies in the space

$$P := \{p \in \pi_{2r-1} \mid (0, \tfrac{1}{2}): p^{(j)}(0) = 0, 0 \leqslant j \leqslant r-1, j \neq s\},$$

where $\pi_{2r-1}$ denotes polynomials of degree $2r-1$. It is easily seen that $\dim P = r+1$. However the $r+2$ functions $\{\psi_s(.-i): -r-1 \leqslant i \leqslant 0\}$ all have supports overlapping $(0, \frac{1}{2})$ and their restrictions to $(0, \frac{1}{2})$ must be linearly dependent.

THEOREM 3.4. *Any function f in $V_1$ can be written uniquely in the form*

$$f = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} b_i^{(s)} B_s(.-i) + \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \psi_s(.-i), \tag{3.8}$$

*for sequences $(b_i^{(s)})_{i=-\infty}^{\infty}$ and $(c_i^{(s)})_{i=-\infty}^{\infty}$ in $l^2$. Moreover if $f(x)$ decays exponentially as $|x| \to \infty$, then $b_i^{(s)}$ and $c_i^{(s)}$ decay exponentially as $|i| \to \infty$.*

*Proof.* First suppose that $f$ has support on $[a, b]$. Let $F$ be the function in $\zeta_{4r-1,r}(\frac{1}{2}Z)$ which vanishes on $(-\infty, a)$ and satisfies $F^{(2r)} = f$. Then $F$ coincides on $(b, \infty)$ with a polynomial $p$ of degree $2r-1$. By Schoenberg's theory [13] there is a unique element $S$ of $\zeta_{4r-1,r}(Z)$ which interpolates $F$ with multiplicity $r$ on $Z$. Since $F-S$ is in $\zeta_{4r-1,r}(\frac{1}{2}Z)$ and has zeros of multiplicity $r$ on $Z$, we have $F = S + \Psi$ form some $\Psi$ in $U$.

Since $F$ vanishes on $(-\infty, a)$ Schoenberg's theory shows that $S(x)$ decays exponentially as $x \to -\infty$. Also $S-p$ interpolates $F-p$ with multiplicity $r$ on $Z$ and, since $F-p$ vanishes on $(b, \infty)$, $S(x)-p(x)$ decays exponentially as $x \to \infty$. Writing $S$ in terms of $B$-splines, we see that $S^{(2r)}(x)$ decays exponentially as $x \to -\infty$ and, since $S^{(2r)}(x) = (S-p)^{(2r)}(x)$, it also decays exponentially as $x \to \infty$. Thus we can write

$$S^{(2r)} = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} b_i^{(s)} B_s(.-i), \tag{3.9}$$

where $b_i^{(s)}$ decays exponentially as $|i| \to \infty$.

Now $\Psi = F - S$ which equals $-S$ on $(-\infty, a)$ and equals $p - S$ on $(b, \infty)$. Thus $\Psi(x)$ decays exponentially as $|x| \to \infty$. Applying Theorem 3.1 and differentiating $2r$ times then gives

$$\Psi^{(2r)} = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \psi_s(.-i), \tag{3.10}$$

where $c_i^{(s)}$ decays exponentially as $i \to \infty$. Adding (3.9) and (3.10) gives (3.8).

In particular, we can write for $j = 0, ..., r-1$, $k \in Z$,

$$B_j(2x-k) = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} b_{2i-k,j}^{(s)} B_s(x-i) + \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_{2i-k,j}^{(s)} \psi_s(x-i), \qquad x \in \mathbf{R}, \tag{3.11}$$

where for some $K>0$, $0<\lambda<1$,

$$|b_{i,j}^{(s)}| \leqslant K\lambda^{|i|}, \qquad |c_{i,j}^{(s)}| \leqslant K\lambda^{|i|}, \qquad s=0, ..., r-1, i \in Z. \tag{3.12}$$

Now any function $f$ in $V_1$ can be written

$$f(x) = \sum_{j=0}^{r-1} \sum_{k=-\infty}^{\infty} a_k^{(j)} B_j(2x-k), \qquad x \in \mathbf{R}, \tag{3.13}$$

where for $j=0, ..., r-1$, $a_j = (a_k^{(j)})_{k=-\infty}^{\infty}$ lies in $l^2$ with

$$\|a_j\|_2 \leqslant C \, \|f\|_2 \tag{3.14}$$

for some constant $C$. Then (3.11) and (3.13) give (3.8), where

$$b_i^{(s)} = \sum_{j=0}^{r-1} \sum_{k=-\infty}^{\infty} a_k^{(j)} b_{2i-k,j}^{(s)}, \tag{3.15}$$

$$c_i^{(s)} = \sum_{j=0}^{r-1} \sum_{k=-\infty}^{\infty} a_k^{(j)} c_{2i-k,j}^{(s)}. \tag{3.16}$$

It follows easily from (3.12), (3.14), (3.15), and (3.16) that for $s=0, ..., r-1$, the sequences $b_s := (b_i^{(s)})_{i=-\infty}^{\infty}$ and $c_s := (c_i^{(s)})_{i=-\infty}^{\infty}$ are in $l^2$ and

$$\|b_s\|_2 \leqslant A \, \|f\|_2, \qquad \|c_s\|_2 \leqslant A \, \|f\|_2, \tag{3.17}$$

for some constant $A$. If $f(x)$ decays exponentially as $|x| \to \infty$, then for $j=0, ..., r-1$, we see from (3.13) that $a_k^{(j)}$ decays exponentially as $|k| \to \infty$ and again it follows from (3.12), (3.15), and (3.16) that $b_i^{(s)}$ and $c_i^{(s)}$ decay exponentially as $|i| \to \infty$. ∎

COROLLARY 3.2. *The functions* $\{\psi_s(.-i): i \in Z, s=0, ..., r-1\}$ *form a Riesz basis for* $W$.

*Proof.* Take $f$ in $W$. Then by Theorem 3.4 we can write

$$f = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \psi_s(.-i), \tag{3.18}$$

for a sequence $c_s := (c_i^{(s)})_{i=-\infty}^{\infty}$ in $l^2$. Clearly $\|f\|_2 \leqslant C \sum_{s=0}^{r-1} \|c_s\|_2$ for some constant $C$. Moreover, by (3.17) we have $\sum_{s=0}^{r-1} \|c_s\|_2 \leqslant B \, \|f\|_2$ for some constant $B$, which completes the proof. ∎

COROLLARY 3.3. *For* $s=0, ..., r-1$, *the functions* $\{\psi_s(.-i): i \in Z\}$ *form a Riesz basis for* $W \cap T_s$.

*Proof.* Take $f$ in $W \cap T_s$. By Theorem 3.4 we can express $f$ as in (3.18). Take $0 \leqslant j \leqslant r-1$, $j \neq s$. Then for $k \in Z$, we have

$$0 = f^{(j)}(k) = \sum_{i=k-r-1}^{k-1} c_i^{(j)} \psi_j^{(j)}(k-i),$$

and so

$$\sum_{i=1}^{r+1} c_{k-i}^{(j)} \psi_j^{(j)}(i) = 0, \qquad k \in Z. \tag{3.19}$$

If we had $\psi_j^{(j)}(i) = 0$, $i = 1, ..., r+1$, then $\Psi_j$ would satisfy the zero interpolation conditions for the solvable problem (3.1), which contradicts $\Psi_j$ having compact support. Thus the sequence $c_j := (c_i^{(j)})_{i=-\infty}^{\infty}$ satisfies the non-trivial recurrence relation (3.19) and, since $c_j$ is in $l^2$, we must have $c_i^{(j)} = 0$, $i \in Z$.

Since this holds for all $j$ with $0 \leqslant j \leqslant r-1$, $j \neq s$, (3.18) becomes

$$f = \sum_{i=-\infty}^{\infty} c_i^{(s)} \psi_s(. - i).$$

It follows from Corollary 3.2 that $\{\psi_s(. - i): i \in Z\}$ forms a Riesz basis for $W \cap T_s$. ∎

## 4. An Example

We now consider the simplest case $r = 2$ and express the functions $\psi_0$ and $\psi_1$ (up to normalisation) in terms of the wavelets $f_1$ and $g_1$ of Theorem 5.1 of [7]. For completeness we first give the construction of $f_1$ and $g_1$.

Let $N_0^7$ be the usual $B$-spline of degree 7 with double knots at 0, ..., 3 and a singe knot at 4. Let $N_1^7$ be the corresponding $B$-spline with a single knot at 0 and double knots at 1, ..., 4, so that $N_1^7(x) = N_0^7(4-x)$. The remaining $B$-splines $N_i^7$, for integers $i$, are given by $N_{i+2}^7(x) = N_i^7(x-1)$. We define a function $F$ by

$$F_{i,0}(x) = N_i^7(2x) + N_{5-i}^7(2x), \qquad i = 0, 1, 2,$$

$$F_{i,1}(x) = F_{i,0}(x)\, F_{i+1,0}(1) - F_{i+1,0}(x)\, F_{i,0}(1), \qquad i = 0, 1, \tag{4.1}$$

$$F(x) = F_{0,1}(x)\, F'_{1,1}(1) - F_{1,1}(x)\, F'_{0,1}(1). \tag{4.2}$$

A function $G$ is defined by

$$G_{i,0}(x) = N_i^7(2x) - N_{5-i}^7(2x), \qquad i = 0, 1, 2,$$

and (4.1), (4.2) with $F$ replaced throughout by $G$. We now define

$$f_1 = F^{(4)}, \qquad g_1 = G^{(4)}.$$

Then $f_1$ and $g_1$ lie in $W$ with support on $[0, 3]$ and are, respectively, even and odd about $\frac{3}{2}$.

THEOREM 4.1. *The functions* $\tilde{\psi}_0$, $\tilde{\psi}_1$ *defined by*

$$\tilde{\psi}_0(x) = g_1'(1)(f_1(x) + f_1(x-1)) - f_1'(1)(g_1(x) - g_1(x-1)), \tag{4.3}$$

$$\tilde{\psi}_1(x) = g_1(1)(f_1(x) - f_1(x-1)) - f_1(1)(g_1(x) + g_1(x-1)), \tag{4.4}$$

*are non-zero constant multiples of* $\psi_0$, $\psi_1$, *respectively.*

*Proof.* Since $\tilde{\psi}_0$, $\tilde{\psi}_1$ lie in $W$ with support in $[0, 4]$, it is sufficient to show that they do not vanish identically and

$$\tilde{\psi}_0'(k) = \tilde{\psi}_1(k) = 0, \qquad k = 1, 2, 3. \tag{4.5}$$

By the symmetry properties of $f_1$ and $g_1$ we see that $\tilde{\psi}_0$ and $\tilde{\psi}_1$ are, respectively, symmetric and anti-symmetric about 2. So (4.5) is satisfied for $k = 2$. From (4.3) and (4.4) we see that (4.5) is satisfies for $k = 1$, and so by symmetry it is also satisfied for $k = 3$.

Now if $f_1'(1) = 0$, then $f_1$ lies in $W \cap T_0$ and has support on $[0, 3]$, which contradicts Theorem 3.2. Now it follows from Theorems 4.2 and 5.1 of [7] that $f_1$, $f_1(. - 1)$, $g_1$, $g_1(. - 1)$ are linearly independent. Since $f_1'(1) \neq 0$, we see from (4.3) that $\tilde{\psi}_0$ does not vanish identically. Similarly we can show $f_1(1) \neq 0$ and deduce from (4.4) that $\tilde{\psi}_1$ does not vanish identically. ∎

## REFERENCES


1. G. BATTLE, A block spin construction of ondelettes, Part 1: Lemarie functions, *Comm. Math. Phys.* **110** (1987), 601–615.
2. C. K. CHUI, "An Introduction to Wavelets," Academic Press, San Diego, 1992.
3. C. K. CHUI AND J. Z. WANG, A general framework for compactly supported splines and wavelets, *J. Approx. Theory*, to appear.
4. I. DAUBECHIES, The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Inform. Theory* **36** (1990), 961–10005.
5. T. N. T. GOODMAN AND S. L. LEE, The Budan–Fourier theorem and Hermite–Birkhoff spline interpolation, *Trans. Amer. Math. Soc.* **271** (1982), 451–467.
6. T. N. T. GOODMAN, S. L. LEE, AND W. S. TANG, Wavelets in wandering subspaces, *Trans. Amer. Math. Soc.* **338** (1993), 639–654.
7. T. N. T. GOODMAN AND S. L. LEE, "Wavelets of Multiplicity $r$," Report AA/921, University of Dundee, 1992.
8. R.-Q. JIA AND C. A. MICCHELLI, Using the refinement equation for the construction of prewavelets II: Powers of two, *in* "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 209–246, Academic Press, New York, 1991.

9. S. L. LEE, *B*-splines for cardinal Hermite interpolation, *Linear Algebra Appl.* **12** (1975), 269–280.
10. G. G. LORENTZ, K. JETTER, AND S. D. RIEMENSCHNEIDER, "Birkhoff Interpolation," Addison–Wesley, Reading, MA, 1983.
11. S. G. MALLAT, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$, *Trans. Amer. Math. Soc.* **315** (1989), 69–87.
12. Y. MEYER, "Ondelettes et Opérateurs I: Ondelettes," Hermann, Paris, 1990.
13. I. J. SCHOENBERG, "Cardinal Spline Interpolation," Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1973.
14. I. J. SCHOENBERG AND A. SHARMA, Cardinal interpolation and spline functions. V. The *B*-splines for cardinal Hermite interpolation, *Linear Algebra Appl.* **7** (1973), 1–42.